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A UNIVERSAL COMPLETION OF THE *ZX-calculus*

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A universal completion of the ZX-calculus

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Abstract

In this paper, we give a universal completion of the ZX-calculus for the whole of pure qubit quantum mechanics. This proof is based on the completeness of another graphical language: the ZW-calculus, with direct translations between these two graphical systems.

1 Introduction

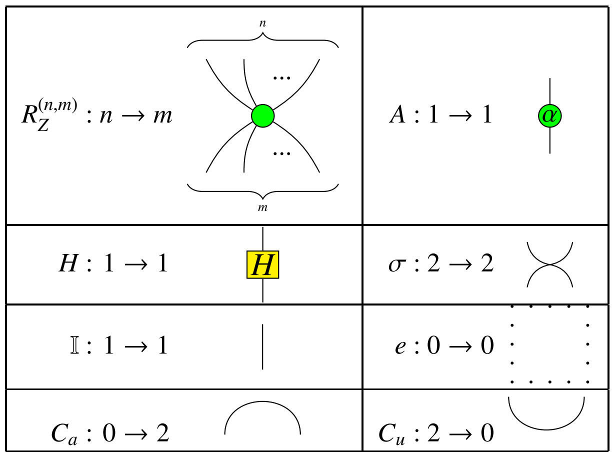
The ZX-calculus introduced by Coecke and Duncan [1] is an intuitive yet mathematically strict graphical language for quantum computing: it is formulated within the framework of compacted closed categories which has a rigorous underpinning for graphical calculus [2], meanwhile being an important branch of categorical quantum mechanics (CQM) pioneered by Abramsky and Coecke [3]. Notably, it has intuitional and simple rewriting rules to transform diagrams from one to another. Each diagram in the ZX-calculus has a standard interpretation in the Hilbert spaces, thus makes it relevant for quantum computing. For the past ten years, the ZX-calculus has enjoyed sucess in applying to fields of quantum information and quantum computation (QIC), in particular (topological) measurement-based quantum computing [4, 5] and quantum error correction [6, 7].

To realise its greatest advantage, the so-called completeness is of concerned with the ZX-calculus: any equation of diagrams that holds true under the standard interpretation in Hilbert spaces can be derived diagrammatically. It has been shown in [8] that the original version of the ZX-calculus [1] plus the Euler decomposition of Hadamard gate is incomplete for the overall pure qubit quantum mechanics (QM). Since then, plenty of efforts have been devoted to completion of some part of QM: real QM [9], stabilizer QM [10], single qubit Clifford+T QM [11] and Clifford+T QM [12]. Amongst them, the completeness of ZX-calculus for Clifford+T QM is especially interesting, since it is approximatively universal for QM. Note that their proof relies on the completeness of ZW-calculus for ”qubits with integer coefficients”.

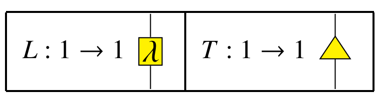
In this paper, we prove that the ZX-calculus is complete for the overall pure qubit QM. Our proof is based on the completeness of ZW-calculus for the whole qubit QM [13]: we first introduce a triangle and a series of λ-labeled boxes ($\lambda \geq 0$), which turns out to be expressible in ZX-calculus without these symbols. Then we establish reversible translations from ZX to ZW and vice versa. By checking carefully that all the ZW rewriting rules still hold under translation from ZW to ZX, we finally finished the proof of completeness of ZX-calculus for the overall qubit QM.

2 ZX-calculus

The ZX-calculus is a compact closed category $\mathfrak{C}$. The objects of $\mathfrak{C}$ are natural numbers: $0, 1, 2, \cdots$; the tensor of objects is just addition of numbers: $m \otimes n = m + n$. The morphisms of $\mathfrak{C}$ are diagrams of the ZX-calculus. A general diagram $D : k \to l$ with k inputs and l outputs is generated by:



where $m, n \in \mathbb{N}$, $\alpha \in [0, 2\pi)$, and e represents an empty diagram. For the purposes of this paper we extend the language with two new symbols (although in principle they could be eliminated, see lemma 2.1):



where $\lambda \geq 0$.

The composition of morphisms is to combine these components in the following two ways: for any two morphisms $D_1 : a \to b$ and $D_2 : c \to d$, a *paralell composition* $D_1 \otimes D_2 : a + c \to b + d$ is obtained by placing D_1 and D_2 side-by-side with D_1 on the left of D_2; for any two morphisms $D_1 : a \to b$ and $D_2 : b \to c$, a *sequential composition* $D_2 \circ D_1 : a \to c$ is obtained by placing D_1 above D_2, connecting the outputs of D_1 to the inputs of D_2.

There are two kinds of rules for the morphisms of $\mathfrak{C}$: the structure rules for $\mathfrak{C}$ as an compact closed category, as well as original rewriting rules listed in Figure 1 and our extended rules listed in Figure 2 and Figure 3.

Note that all the diagrams should be read from top to bottom.

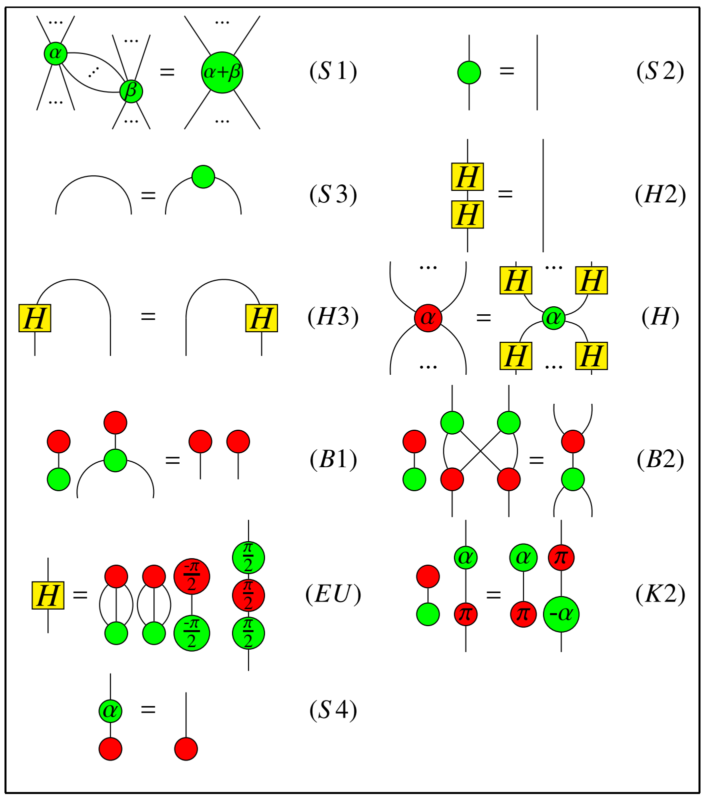


Figure 1: Original ZX-calculus rules, where $\alpha, \beta \in [0,\ 2\pi)$.

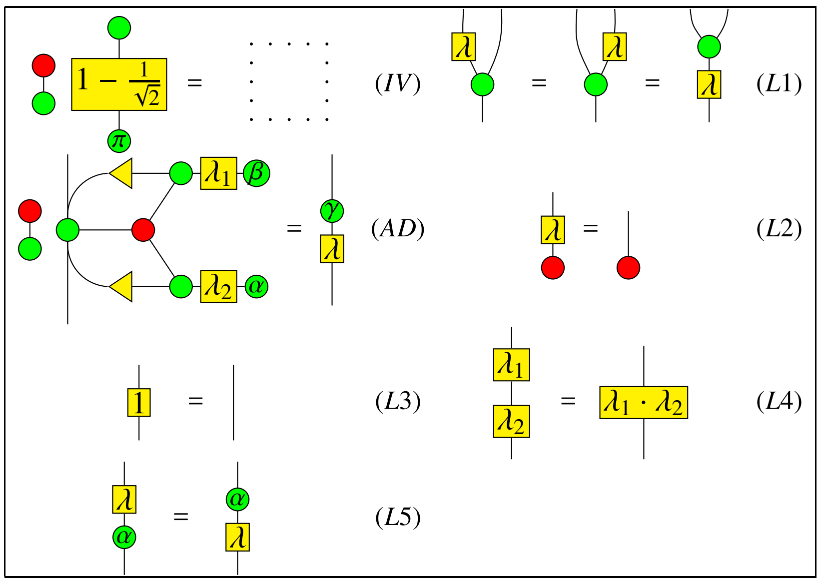


Figure 2: Extended ZX-calculus rules for λ and addition, where $\lambda, \lambda_1, \lambda_2 \geq 0, \alpha, \beta, \gamma \in [0,\ 2\pi)$; in (AD), $\lambda e^{i\gamma} = \lambda_1 e^{i\beta} + \lambda_2 e^{i\alpha}$.

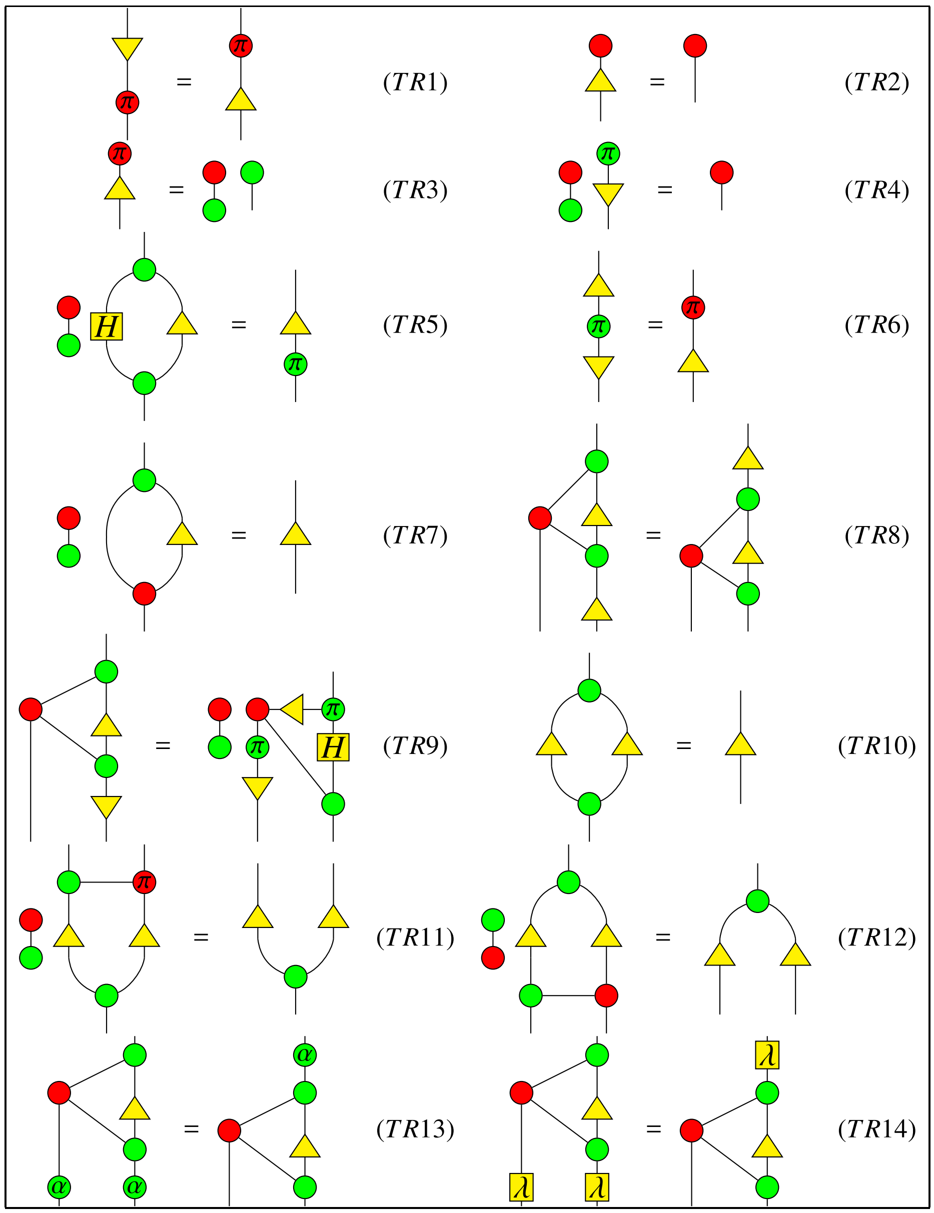
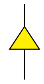
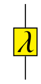
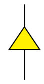
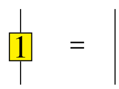


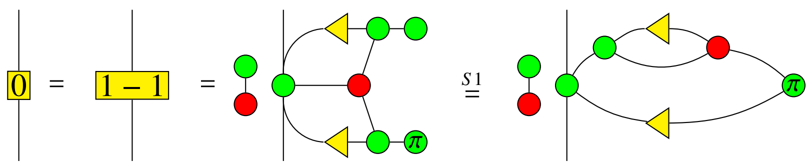
Figure 3: Extended ZX-calculus rules for triangle, where $\lambda \geq 0, \alpha \in [0,\ 2\pi)$.

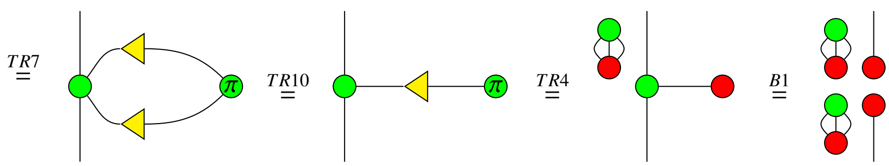
Lemma 2.1 *The triangle*  *and the lambda box*  *are expressible in Z and X phases.*

Proof: The triangle  has been represented by ZX phases in [12]. So we only need to deal with the lambda box. First we can write λ as a sum of its integer part and remainder part: $\lambda = [\lambda] + \{\lambda\}$, where $[\lambda]$ is a non-negative integer and $0 \leq \{\lambda\} < 1$. Let $n = [\lambda]$. If $n = 1$, then by rule (L3),

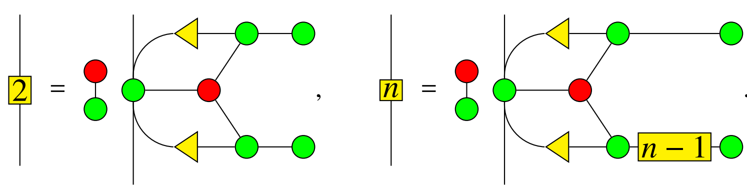


If $n = 0$, then

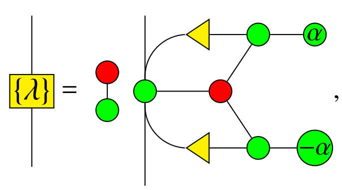




If $n \geq 2$, then by rule (AD) and induction, we have

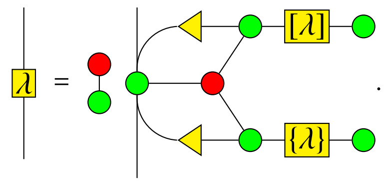


Again by rule (AD), we have



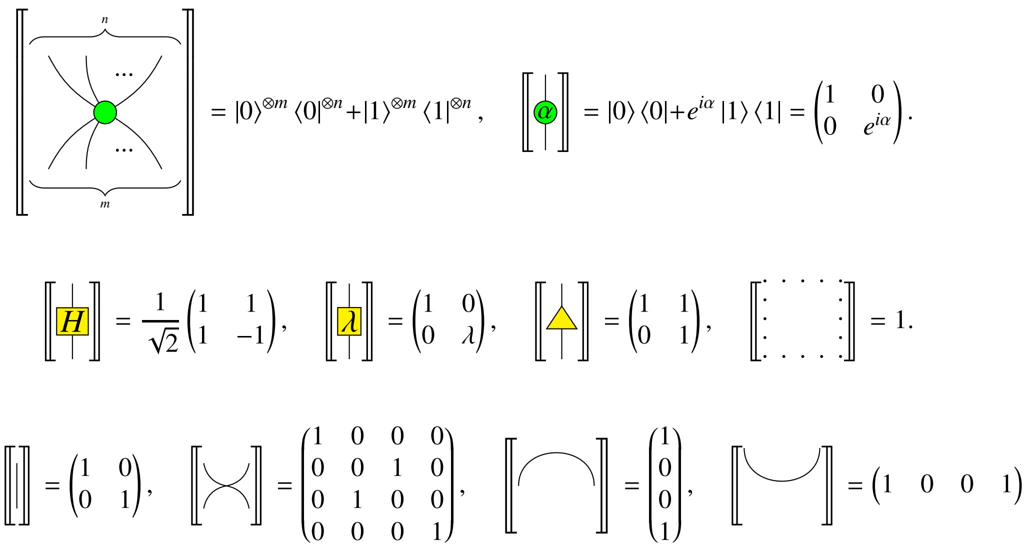
where $\alpha = arccos\frac{\{\lambda\}}{2}$.

Therefore, we have



□

The diagrams in the ZX-calculus have a standard interpretation $[\![\cdot]\!]$ in the category of Hilbert spaces:

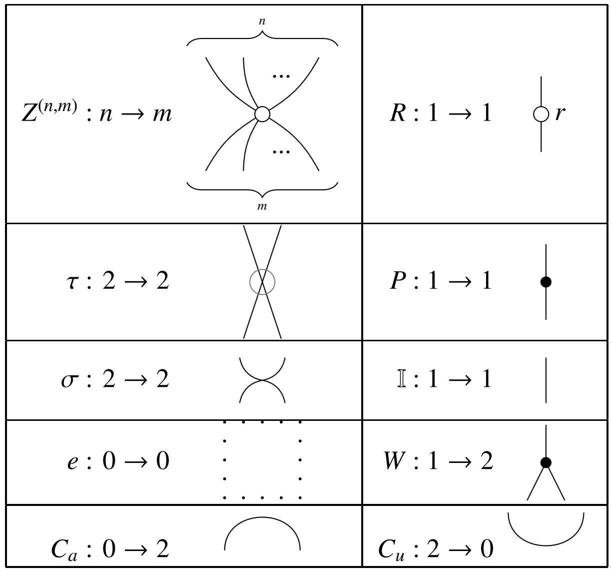


$$[\![D_1 \otimes D_2]\!] = [\![D_1]\!] \otimes [\![D_2]\!], \quad [\![D_1 \circ D_2]\!] = [\![D_1]\!] \circ [\![D_2]\!].$$

It can be verified that the interpretation $[\![\cdot]\!]$ is a monoidal functor.

3 ZW-calculus

The ZW-calculus is a compact closed category $\mathfrak{F}$. The objects of $\mathfrak{F}$ are natural numbers: $0, 1, 2, \cdots$; the tensor of objects is just addition of numbers: $m \otimes n = m + n$. The morphisms of $\mathfrak{F}$ are diagrams of the ZX-calculus. A general diagram $D : k \to l$ with k inputs and l outputs is generated by:



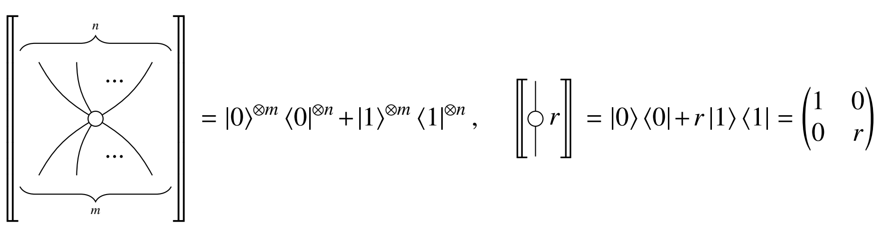
where $m, n \in \mathbb{N}$, $r \in \mathbb{C}$, and e represents an empty diagram.

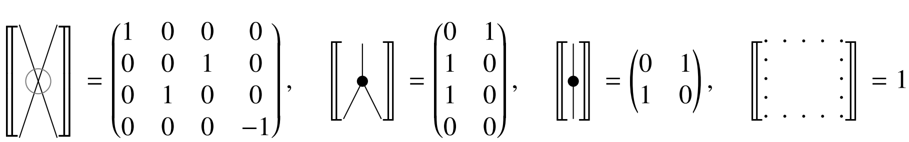
The composition of morphisms is to combine these components in the following two ways: for any two morphisms $D_1 : a \to b$ and $D_2 : c \to d$, a *paralell composition* $D_1 \otimes D_2 : a + c \to b + d$ is obtained by placing D_1 and D_2 side-by-side with D_1 on the left of D_2; for any two morphisms $D_1 : a \to b$ and $D_2 : b \to c$, a *sequential composition* $D_2 \circ D_1 : a \to c$ is obtained by placing D_1 above D_2, connecting the outputs of D_1 to the inputs of D_2.

There are two kinds of rules for the morphisms of $\mathfrak{F}$: the structure rules for $\mathfrak{F}$ as an compact closed category, as well as the rewriting rules listed in Figure 4, 5, 6, **??**.

Note that all the diagrams should be read from top to bottom.

The diagrams in the ZX-calculus have a standard interpretation $[\![\cdot]\!]$ in the category of Hilbert spaces:





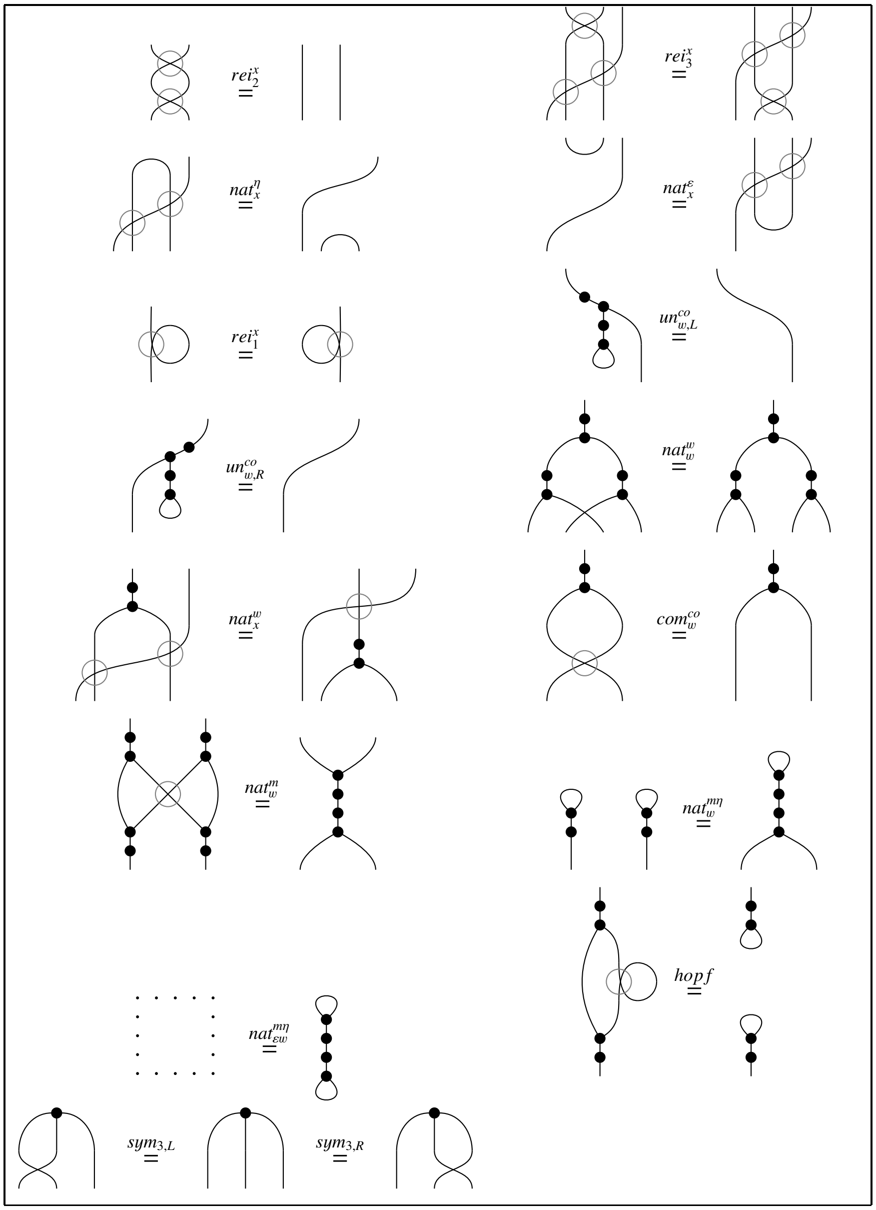


Figure 4: ZW-calculus rules I

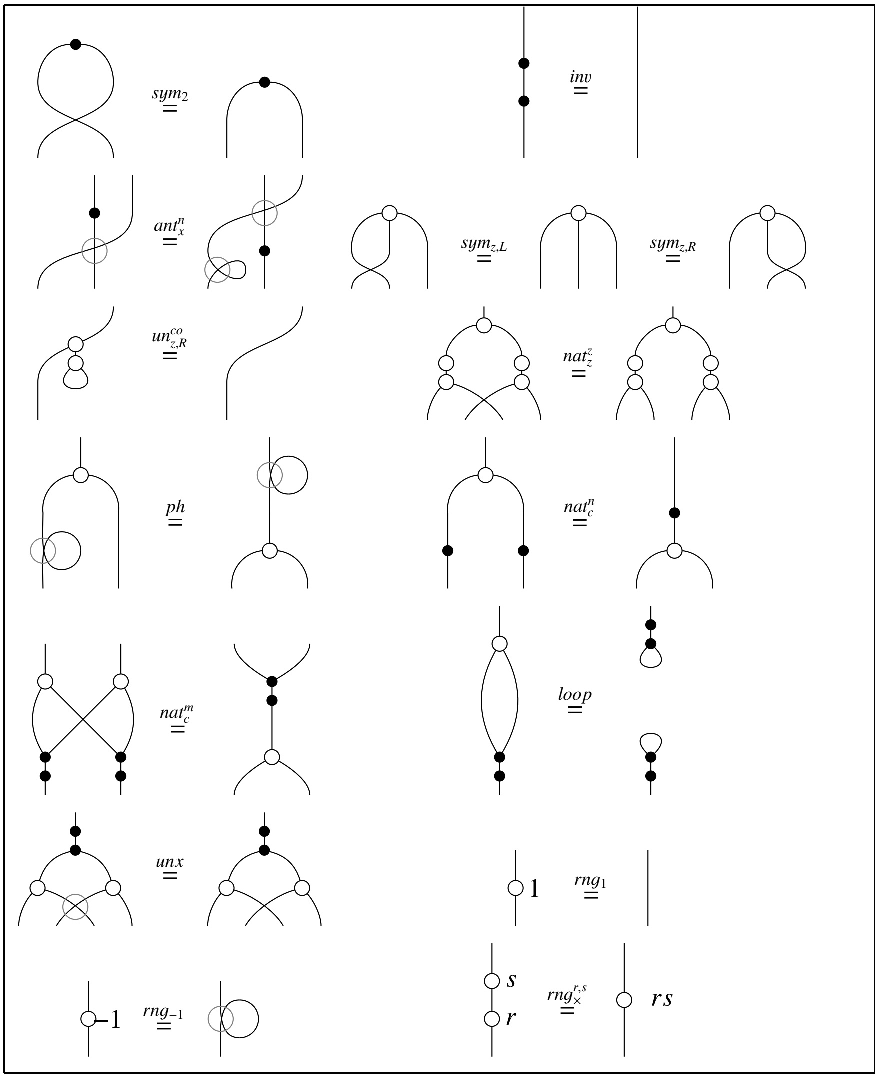


Figure 5: ZW-calculus rules II

$$[\![D_1 \otimes D_2]\!] = [\![D_1]\!] \otimes [\![D_2]\!], \quad [\![D_1 \circ D_2]\!] = [\![D_1]\!] \circ [\![D_2]\!].$$

It can be verified that the interpretation $[\![\cdot]\!]$ is a monoidal functor.

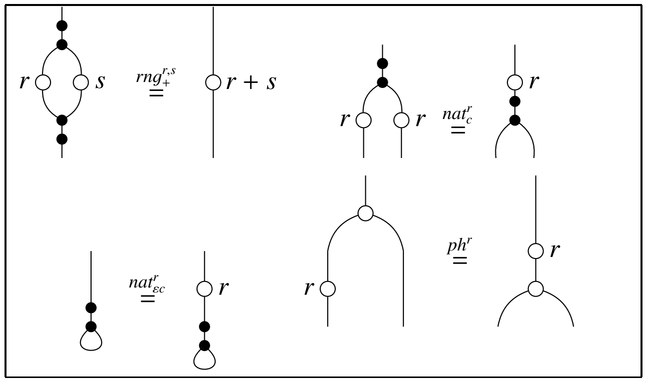
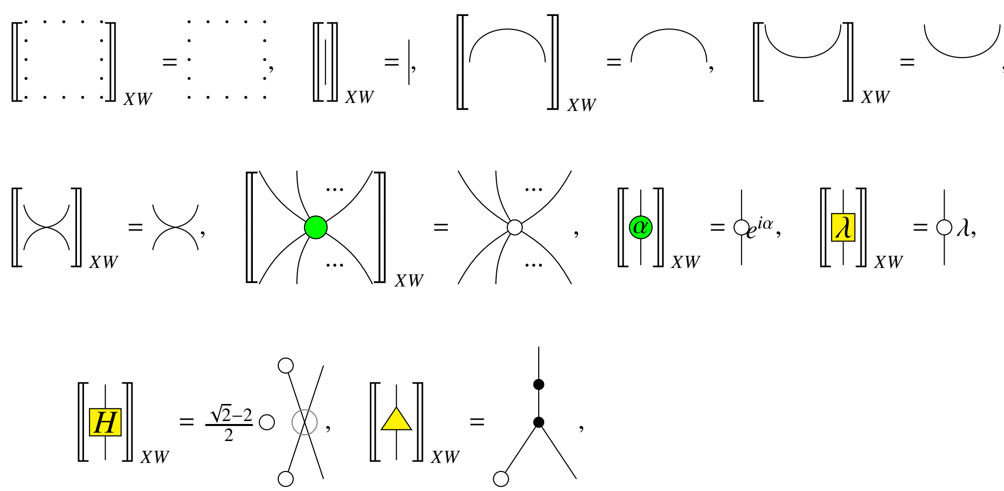


Figure 6: ZW-calculus rules III

4 Interpretations from ZX-calculus to ZW-calculus and back forth

First we define the interpretation $[\![\cdot]\!]_{XW}$ from ZX-calculus to ZW-calculus as follows:



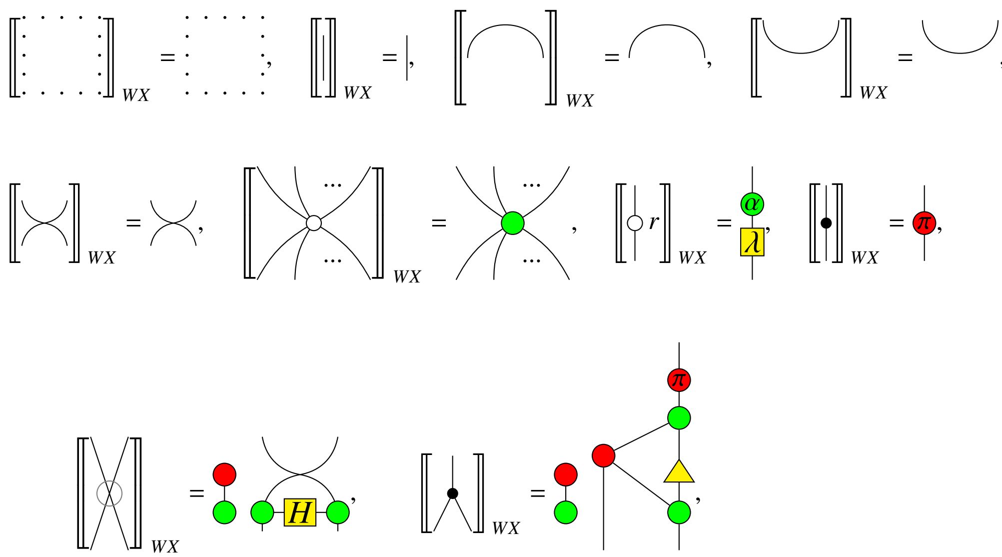
$$[\![D_1 \otimes D_2]\!]_{XW} = [\![D_1]\!]_{XW} \otimes [\![D_2]\!]_{XW}, \quad [\![D_1 \circ D_2]\!]_{XW} = [\![D_1]\!]_{XW} \circ [\![D_2]\!]_{XW},$$

where $\alpha \in [0,\ 2\pi),\ \lambda \geq 0$.

Lemma 4.1 *Suppose D is an arbitrary diagram in ZX-calculus. Then $[\![[\![D]\!]_{XW}]\!] = [\![D]\!]$.*

The proof is easy.

Next we define the interpretation $[\![\cdot]\!]_{WX}$ from ZW-calculus to ZX-calculus as follows:



$$[\![D_1 \otimes D_2]\!]_{WX} = [\![D_1]\!]_{WX} \otimes [\![D_2]\!]_{WX}, \quad [\![D_1 \circ D_2]\!]_{WX} = [\![D_1]\!]_{WX} \circ [\![D_2]\!]_{WX}.$$

where $r = \lambda e^{i\alpha},\ \alpha \in [0,\ 2\pi),\ \lambda \geq 0$.

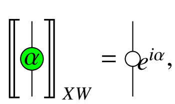
Lemma 4.2 *Suppose D is an arbitrary diagram in ZW-calculus. Then* $[\![[\![D]\!]_{WX}]\!] = [\![D]\!]$.

The proof is easy.

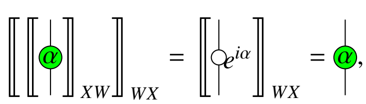
Lemma 4.3 *Suppose D is an arbitrary diagram in ZX-calculus. Then ZX* $\vdash [\![[\![D]\!]_{XW}]\!]_{WX} = D$.

Proof: By the construction of $[\![\cdot]\!]_{XW}$ and $[\![\cdot]\!]_{WX}$, we only need to prove for D as a generator of ZX-calculus. The first six generators in ZX-calculus are the same as the first six generators in ZW-calculus, so we just check for the last four generators in ZX-calculus.

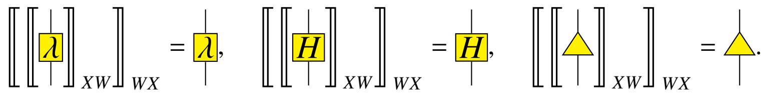
Since



we have



by the definition of $[\![\cdot]\!]_{WX}$ and the ZX rule (L3). Similarly, we can easily check that



□

5 Completeness

Proposition 5.1 *If* $ZW \vdash D_1 = D_2$*, then* $ZX \vdash [\![D_1]\!]_{WX} = [\![D_2]\!]_{WX}$.

Proof: Here we need only to prove that $ZX \vdash [\![D_1]\!]_{WX} = [\![D_2]\!]_{WX}$ where $D_1 = D_2$ is a rewriting rule of ZW-calculus. The whole of Appendix is devoted to prove this proposition. □

Theorem 5.2 *The ZX-calculus is complete for universal pure qubit quantum mechanics: If* $[\![D_1]\!] = [\![D_2]\!]$*, then* $ZX \vdash D_1 = D_2$*,*

Proof: Suppose $D_1, D_2 \in ZX$ and $[\![D_1]\!] = [\![D_2]\!]$. Then by lemma 4.1, $[\![[\![D_1]\!]_{XW}]\!] = [\![D_1]\!] = [\![D_2]\!] = [\![[\![D_2]\!]_{XW}]\!]$. Thus by the completeness of ZW-calculus [13], $ZW \vdash [\![D_2]\!]_{XW} = [\![D_2]\!]_{XW}$. Now by proposition 5.1, $ZX \vdash [\![[\![D_1]\!]_{XW}]\!]_{WX} = [\![[\![D_2]\!]_{XW}]\!]_{WX}$. Finally, by lemma 4.3, $D_1 = D_2$. □

6 Conclusion and further work

In this paper, we show that the ZX-calculus is complete for the universal pure qubit QM, with the aid of completeness of ZW-calculus for the whole qubit QM.

There are several questions for the next step. Firstly, can we derive the completeness of ZX-calculus for Clifford+T QM from the universal completeness? Secondly, can we obtain the completeness of ZX-calculus for stabilizer QM from the universal completeness? Thirdly, can we generalise the completeness result to qudit ZX-calculus for arbitrary dimension d? Furthermore, can we have a proof of completeness that is independent of the ZW-calculus?

It is also interesting to incorporate the rules of the universally complete ZX-calculus in the automated graph rewriting system Quantomatic [16].

Acknowledgement

The authors would like to thank Bob Coecke and Amar Hadzihasanovic for the fruitful discussions and invaluable comments.

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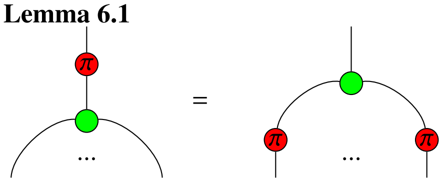
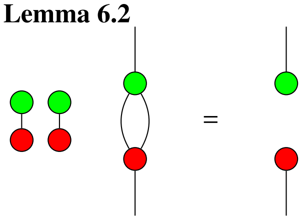
Appendix

Lemma 6.1



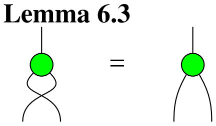
Proof: Proof in [14]. The rules used are $S1$, $S2$, $S3$, H, $H2$, $H3$, $B1$, $B2$, EU, 6.5. □

Lemma 6.2



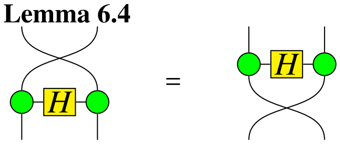
Proof: Proof in [14]. The rules used are $S1$, $S2$, $S3$, $H2$, $H3$, H, $B1$, $B2$. □

Lemma 6.3

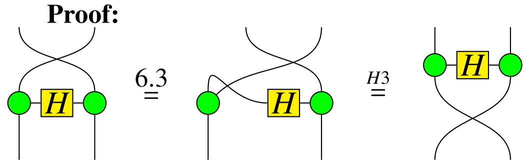


Proof: Proof in [14]. The rules used are 6.2, $B2$, $S2$, $H2$, $S1$, H, $B1$. □

Lemma 6.4

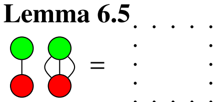


Proof:

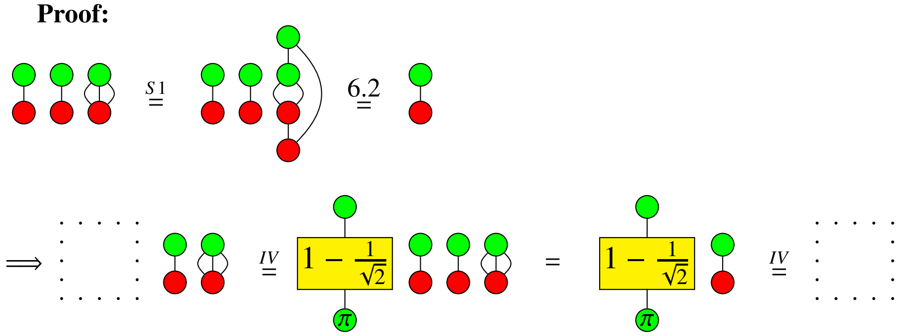


□

Lemma 6.5

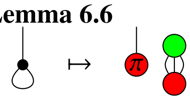


Proof:



□

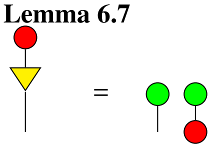
Lemma 6.6



Proof:

Proof in [12] lemma 33. The rules used are 6.2, 6.5, $B1$, $TR2$. □

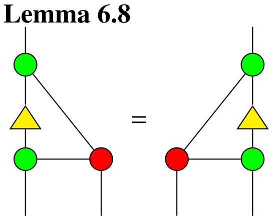
Lemma 6.7



Proof:

Proof in [12] lemma 19. The rules used are 6.2, 6.5, $B1$, $TR2$. □

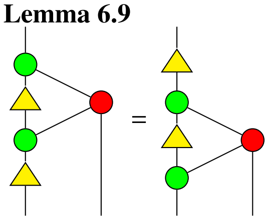
Lemma 6.8



Proof:

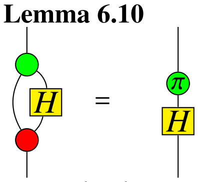
Proof in [12] lemma 26. The rules used are $B2$, $S1$, $TR7$. □

Lemma 6.9

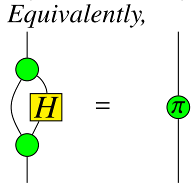


Proof:
Straightforward application of rules 6.3 and $TR8$. □

Lemma 6.10

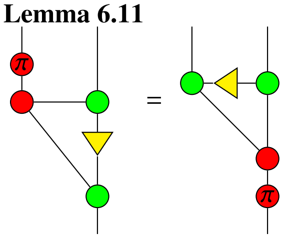


Equivalently,

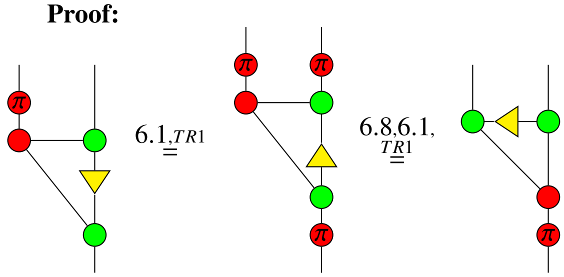


Proof:
Straightforward application of rules H and EU. □

Lemma 6.11

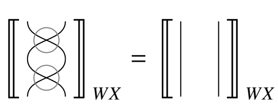


Proof:

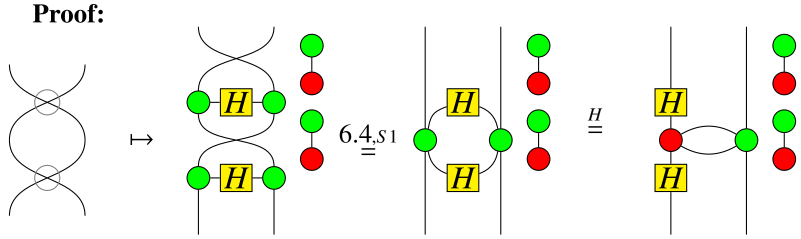


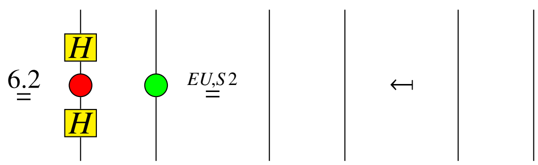
□

Proposition 6.12 *(ZW rule rei_2^x)*

$ZX \vdash$ 

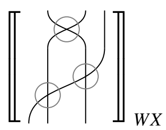
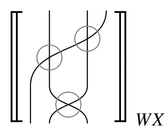
Proof:



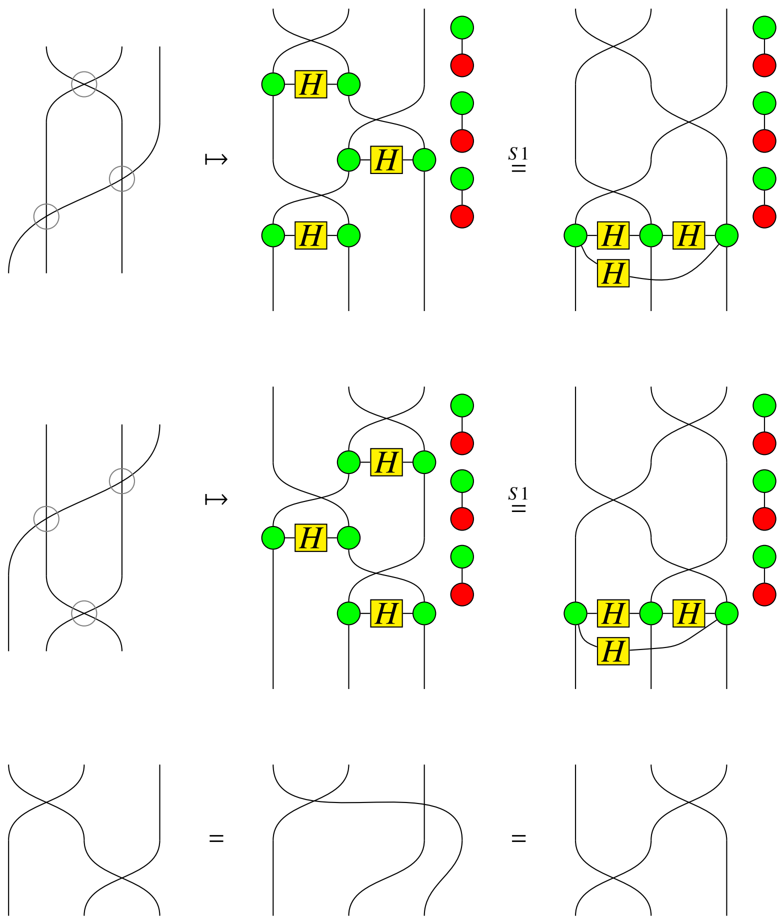


□

Proposition 6.13 *(ZW rule rei_3^x)*

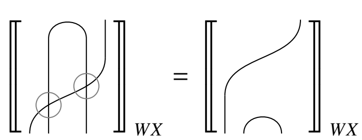
$ZX \vdash$  $=$ 

Proof:

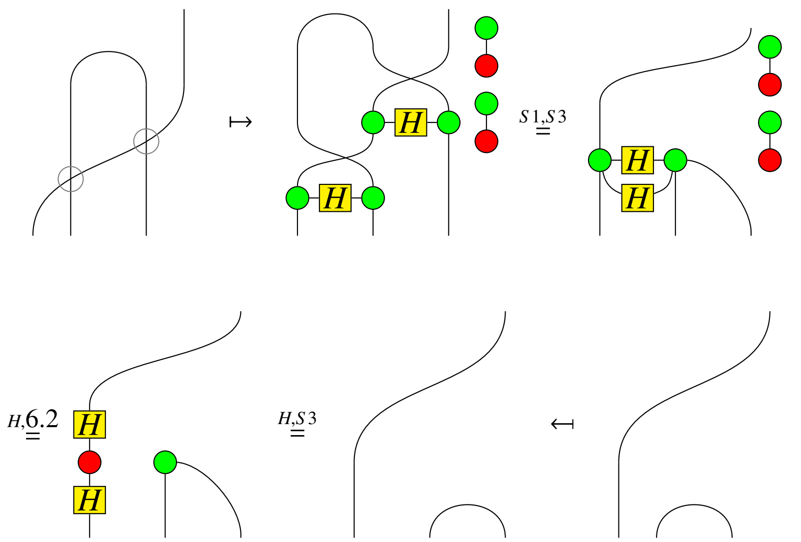


□

Proposition 6.14 *(ZW rule nat_x^η)*

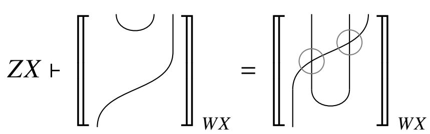
$ZX \vdash$ 

Proof:



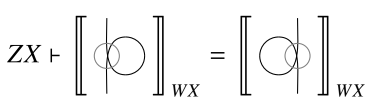
□

Proposition 6.15 *(ZW rule nat^{ε}_{x})*

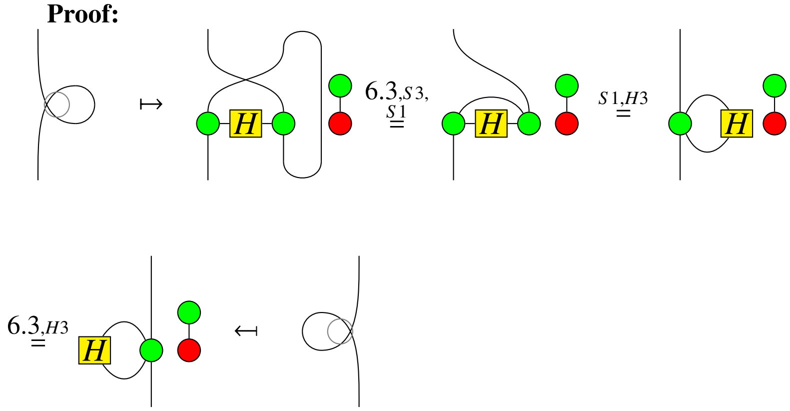


Proof: Similar to proposition 6.14. □

Proposition 6.16 *(ZW rule rei^{x}_{1})*

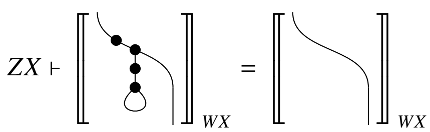


Proof:



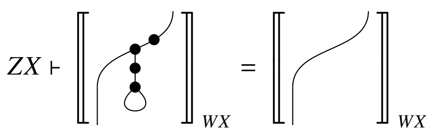
□

Proposition 6.17 *(ZW rule $un^{co}_{w,L}$)*



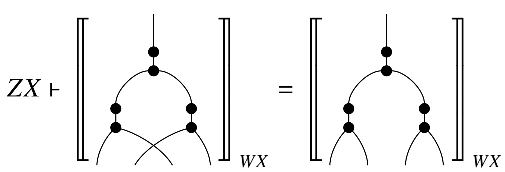
Proof: Similar proof in [12], proposition 7 part 1b. The rules used are 6.5, $S1$, $H2$, H, $S2$, $TR7$, 6.2, $TR2$, 6.7. □

Proposition 6.18 *(ZW rule $un^{co}_{w,R}$)*

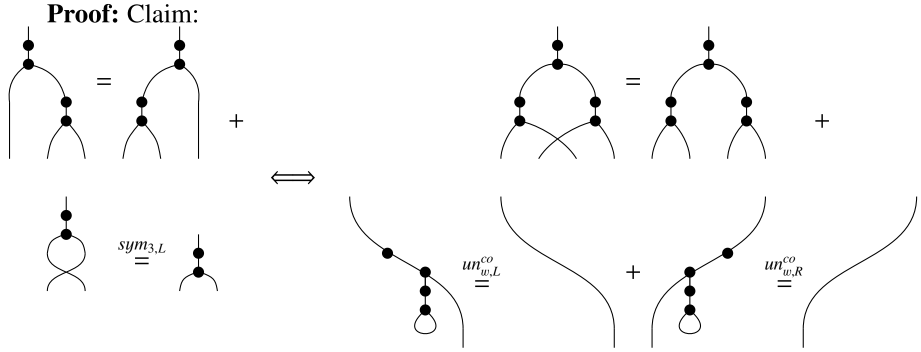


Proof:
Proof in [12], proposition 7 part 1b. The rules used are 6.5, $S1$, $H2$, H, $S2$, $TR7$, 6.2, $TR2$, 6.7. □

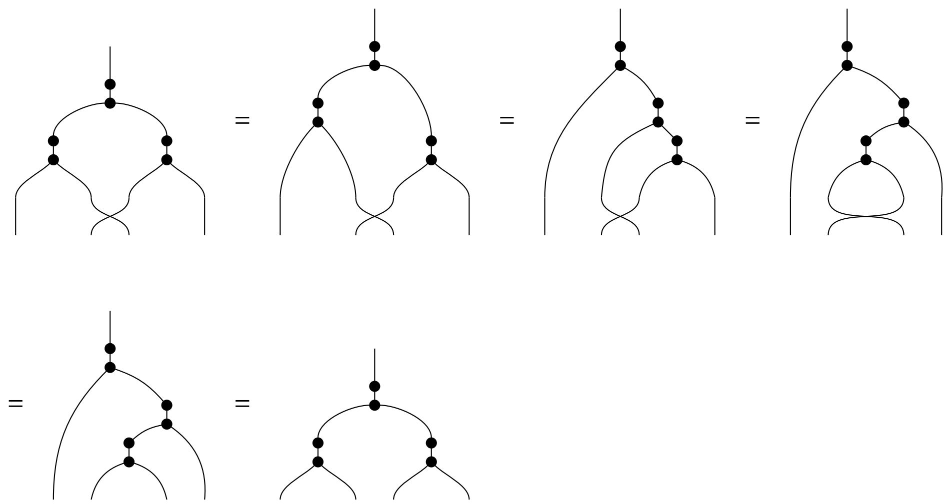
Proposition 6.19 *(ZW rule nat^{w}_{w})*



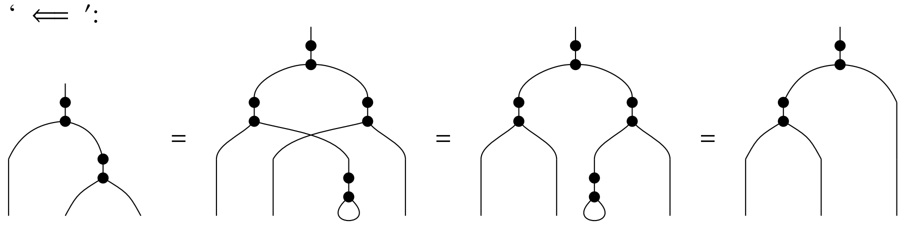
Proof: Claim:



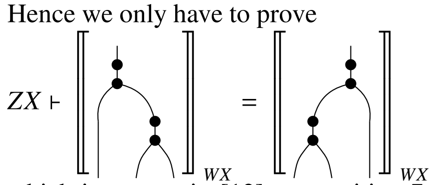
Proof of claim:
‘ $\Longrightarrow$ ’:



‘ ⟸ ’:

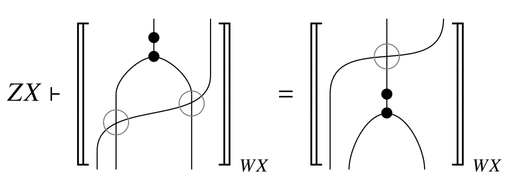


Hence we only have to prove



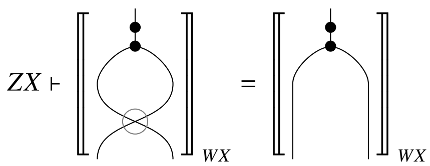
which is proven in [12], proposition 7 part 1a. The rules used are $S1$, $B2$, 6.8, 6.9, $TR8$. □

Proposition 6.20 *(ZW rule nat_x^w)*

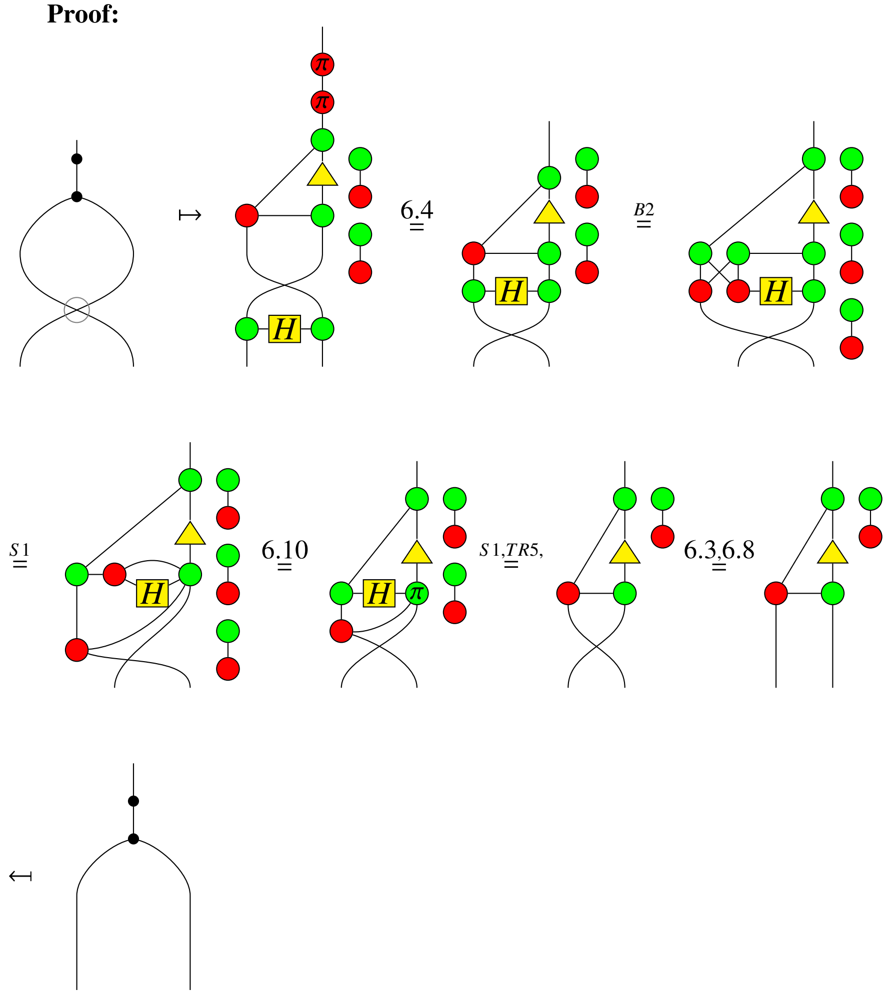


Proof: Proof in [12], proposition 7 part 7a. The rules used are H, 6.4, $S1$, $B2$. □

Proposition 6.21 *(ZW rule com_w^{co})*

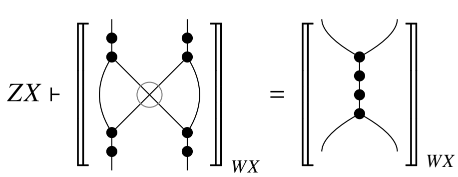


Proof:



□

Proposition 6.22 *(ZW rule nat_w^m)*



Proof: Proof in [12], proposition 7, part 5a. It was proved in eight parts, (i), (ii), (iii), (iv), (v), (vi), (vii), ($viii$).

The rules used in part (i) are $B2$, $S1$, H, $TR5$, 6.1.

The rules used in part (ii) are 6.8, $S1$, $B2$, $S1$, $TR11$, 6.1, $TR1$.

The rules used in part (iii) are $TR6$, 6.8, $S1$, $TR8$, 6.1.

The rules used in part (iv) are 6.8, $S1$, $B2$, 6.1, $TR11$.

The rules used in part (v) are 6.1, part (v).

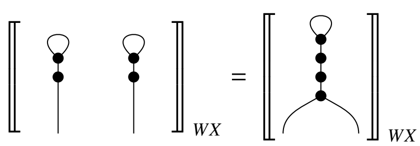
The rules used in part (vi) are part (i), 6.8, $TR9$, $S1$, 6.1, 6.2, $TR1$, $K2$, part (iii), part (v), part (ii).

The rules used in part (vii) are 6.1, $S1$, $TR1$, $B2$, $TR7$, 6.8, $TR11$.

The rules used in part ($viii$) are 6.1, H, $B2$, $S1$, 6.2, 6.10.

Finally, the proof of this proposition uses the rules 6.8, $S1$, part ($viii$), 6.2, part (vii), part (vi), $B2$, 6.3, 6.11, 6.1, $TR1$. □

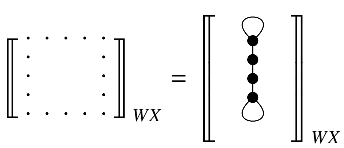
Proposition 6.23 *(ZW rule $nat_w^{m\eta}$)*

$ZX \vdash$ 

Proof:

Proof in [12], proposition 7 part 5c. The rules used are 6.6, $B1$, $TR2$. □

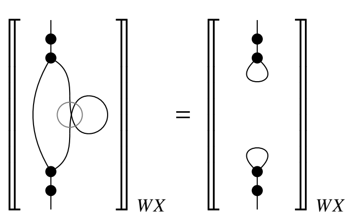
Proposition 6.24 *(ZW rule $nat_{\varepsilon,w}^{m\eta}$)*

$ZX \vdash$ 

Proof:

Proof in [12], proposition 7 part 5c. The rules used are 6.6, $S1$, $B1$, 6.5. □

Proposition 6.25 *(ZW rule $hopf$)*

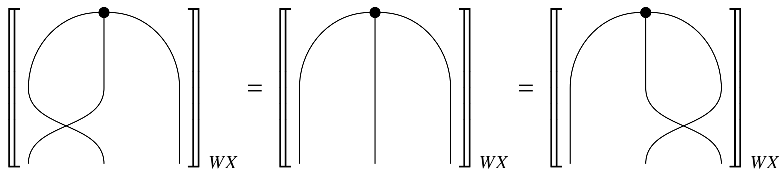
$ZX \vdash$ 

Proof:

Proof in [12], proposition 7 part 5d. The rules used are 6.39, 6.2, $S1$, $TR10$, $TR4$, $B1$.

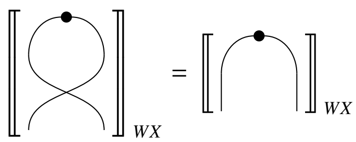
□

Proposition 6.26 *(ZW rules $sym_{3,L}$ and $sym_{3,R}$)*

$ZX \vdash$ 

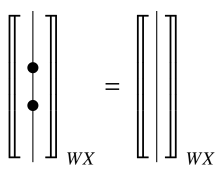
Proof: Proof in [12], proposition 7 part $0b$ and $0b'$ respectively. The rules used are 6.8, 6.3, 6.1, $S1$, $TR1$. □

Proposition 6.27 *(ZW rule sym_2)*

$ZX \vdash$ 

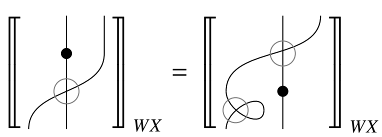
Proof: Proof in [12], proposition 7 part $0a$. The rules used are $S1$, 6.3. □

Proposition 6.28 *(ZW rule inv)*

$ZX \vdash$ 

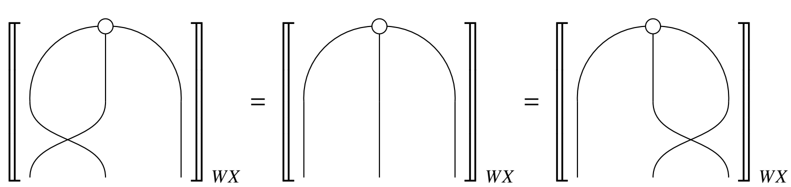
Proof: Proof in [12], proposition 7 part $2a$. The rule used is $S1$. □

Proposition 6.29 *(ZW rule ant_x^η)*

$ZX \vdash$ 

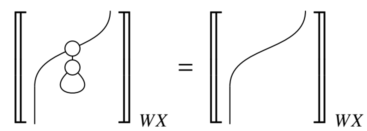
Proof: Proof in [12], proposition 7 part $7b$. The rules used are 6.1, H, $S1$. □

Proposition 6.30 *(ZW rules $sym_{z,L}$ and $sym_{z,R}$)*

$ZX \vdash$ 

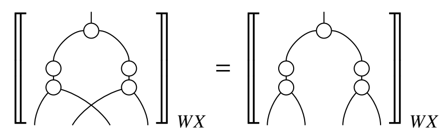
Proof: Proof follows directly from rules $S1$ and 6.3. □

Proposition 6.31 *(ZW rule $un^{co}_{z,R}$)*

$ZX \vdash$ 

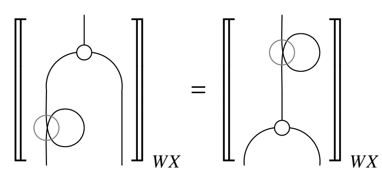
Proof: Proof follows directly from rule $S1$. □

Proposition 6.32 *(ZW rule nat^{z}_{z})*

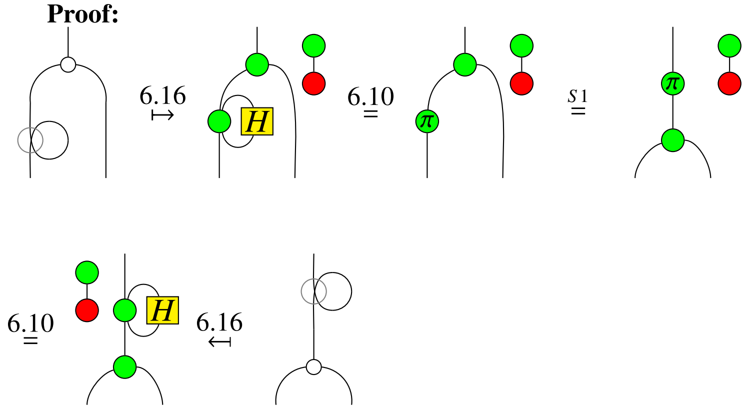
$ZX \vdash$ 

Proof: Proof follows directly from rules $S1$ and 6.3. □

Proposition 6.33 *(ZW rule ph)*

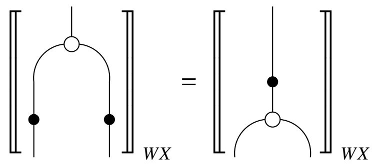
$ZX \vdash$ 

Proof:



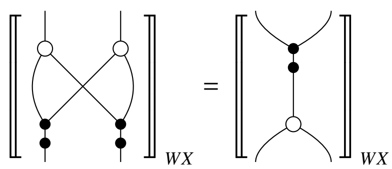
. □

Proposition 6.34 *(ZW rule nat^{m}_{c})*

$ZX \vdash$ 

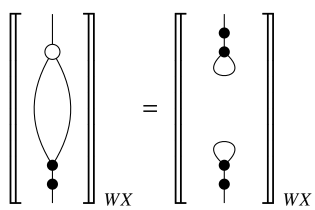
Proof: Proof in [12], proposition 7 part 3a. The rule used is 6.1. □

Proposition 6.35 *(ZW rule nat_c^m)*

$ZX \vdash$ 

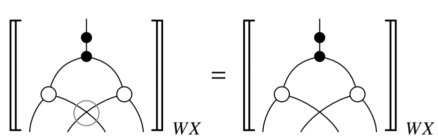
Proof: Proof in [12], proposition 7 part 6a. The rules used are $S1$, $B2$, $TR10$. □

Proposition 6.36 *(ZW rule $loop$)*

$ZX \vdash$ 

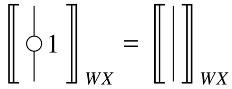
Proof: Proof in [12], proposition 7 part 6c. The rules used are 6.2, $B1$, $TR2$. □

Proposition 6.37 *(ZW rule unx)*

$ZX \vdash$ 

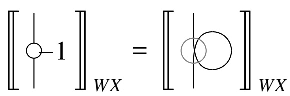
Proof: Proof in [12], proposition 7 part X. The rules used are 6.4, $S1$, $B2$, 6.10, $TR5$. □

Proposition 6.38 *(ZW rule rng_1)*

$ZX \vdash$ 

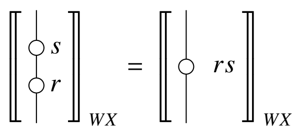
Proof: Proof follows directly from rule $L3$. □

Proposition 6.39 *(ZW rule rng_{-1})*

$ZX \vdash$ 

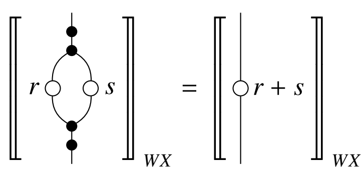
Proof: Similar to 6.33. □

Proposition 6.40 *(ZW rule $rng_{\times}^{r,s}$)*

$ZX \vdash$ 

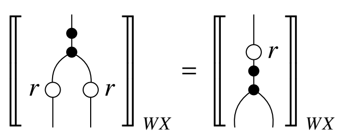
Proof: Proof follows directly from rules $S1$, $L4$, $L5$. □

Proposition 6.41 *(ZW rule $rng_{+}^{r,s}$)*

$ZX \vdash$ 

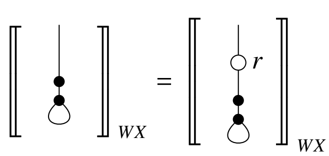
Proof: Proof follows directly from rule AD. □

Proposition 6.42 *(ZW rule nat_{c}^{r})*

$ZX \vdash$ 

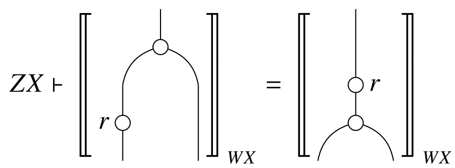
Proof: Proof follows directly from rules $TR13$, $TR14$. □

Proposition 6.43 *(ZW rule $nat_{\epsilon c}^{r}$)*

$ZX \vdash$ 

Proof: Proof follows directly from rules $S4$, $L2$. □

Proposition 6.44 *(ZW rule ph^r)*



Proof: Proof follows directly from rules $S1$, $L1$. □